

# A perturbative moment approach to option pricing.

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## Abstract

In this paper we present a new methodology for option pricing. The main idea consists to represent a generic probability distribution function (PDF) via a perturbative expansion around a given, simpler, PDF (typically a gaussian function) by matching moments of increasing order. Because, as shown in literature, the pricing of path dependent European options can be often reduced to recursive (or nested) one-dimensional integral calculations, the above perturbative moment expansion (PME) leads very quickly to excellent numerical solutions. In this paper, we present the basic ideas of the method and the relative applications to a variety of contracts, mainly: asian, reverse cliquet and barrier options. A comparison with other numerical techniques is also presented.

**Keywords:** Option pricing, Barrier options, Asian options, Reverse Cliquet options, Discrete Monitoring, Quadrature, Gram-Charlier Series, Black-Scholes.

## 1 Introduction

A central problem in today finance is to price correctly and rapidly exotic options. Indeed, in the last years, exotic options have become quite popular and financial institutions traded larger and larger quantity of these sophisticated financial instruments. Moreover, it has become quite common to encounter exotic options embedded in structured bonds, which are addressed also to small investors. There are no limits to market fantasy in the construction of new option varieties. It is therefore of great importance to develop secure and fast methodologies for option pricing. The literature

on this argument is quite huge. Many approaches have been proposed, the most important of which are summarized below:

- (1) **analytical solutions:** usually they are limited to some special cases (simple contracts where the underlying asset is usually assumed to be governed by a log-normal process) [2]. Furthermore these exact solutions cannot be extended to more complex contracts.
- (2) **binomial or trinomial tree:** these popular approaches are extensively used both in academic literature [4, 5] as well as financial institutions because they can be easily adapted to many different situations and are simple from a conceptual point of view. However many drawbacks are present: 1) the convergence is generally slow and the algorithm becomes rapidly inefficient in high dimension (e.g. for barrier options, the problem dimension scales linearly with the number of monitoring dates); 2) the error is not monotonic by increasing the number of nodes [7]; 3) usually the underlying movements are assumed to be described by a log-normal process; 4) the result accuracy depends on how the lattice is defined and last but not least, the method becomes critical for particular choice of market data (e.g. for barrier options, when the spot price is close to barrier level [6]).
- (3) **Monte Carlo methods:** MC is a powerful method for numerical pricing calculation [10, 11, 12]. Indeed when the problem dimension becomes high, it is the natural choice respect to binomial or trinomial tree. That because the pricing error, in MC, scales as  $1/\sqrt{N}$  (where  $N$  is the number of simulations) independently from problem dimension. Moreover, Monte Carlo can be implemented quite straightforward for all path dependent european options, it therefore not surprising that this method is largely used in banks and financial institutions. On the other hand, the convergence speed is low, making MC approach computationally demanding; that have originated a plethora of improved convergence methods and their relatives (reduced variance techniques, quasi Monte Carlo approach based on low discrepancy sequences etc. [12])
- (4) **PDE solution:** as well known, the problem of option pricing can be formulated in terms of partial differential equation (PDE) [2]; therefore a possible strategy, to price derivatives, consists to solve numerically a PDE. A good example of this approach, in the case of path dependent option with discrete sampling, is given in [13, 14, 15]. A strong point in

favor of this method, is the possibility of considering different processes alternative to geometric brownian motion (GBM). However some care must be taken in the implementation of the method.

- (5) **quadrature method:** the option pricing of path dependent european options could be reformulated as a path integral. In the case of discrete fixing dates, this path integral reduces to a multi dimensional integral (whose dimension correspond to the number of observation dates). As shown by some authors [16, 17, 18, 19, 20, 21], often this multiple integral can be further reduced to a series of recursive (nested) one-dimensional integrals (one for each fixing dates), whose solution leads to an estimation of the option price.

The problem faced by quadrature methods is how to treat accurately the density function involved in each of these one-dimensional nested integrals. Three main different approaches have been proposed in literature:

- (i) a basic simple idea [16, 17] consists to discretize, via a grid of points, the density at each time step. Then, by adopting a recursive algorithm, the density function at the next time step is numerically calculated basing on the values at the previous step. This numerical scheme can prove to be expensive, from a computational point of view, as the result accuracy depends directly from the number of points composing the grid.
- (ii) alternative to crude Numerical Recursive Integration (NRI) presented above, one can resort to parametric approximations of the real density, by finding the distribution parameters via quadratic fits. For instance, in the case of asian options, a popular choice is to parametrize the density of underlying arithmetic average, at each observation date, as a log-normal distribution. An improvement was suggested by Lim [19], who proposed a more general mixed density approximation in order to take in account the skewness of the distribution which comes up when high value of underlying volatility are considered.  
However the results, depend strictly on the arbitrary choice of a particular form used to model real densities. In other word, we do not have a rational criteria in order to individuate the better parametric representation.
- (iii) a third alternative consists to combine numerical integration with function approximation. In [18], the authors consider the prob-

lem of a barrier option with discrete monitoring dates. They approximate the solution at each observation date by a Chebyshev polynomials and solve the integral via Gauss-Legendre quadrature. An alternative approach has been proposed by Fusai et al. in [21], again for barrier options. The authors approximate the solution thorough a linear combination of hat functions. All the task consists to relate the coefficients of the linear combination at time  $t_{i+1}$  to the coefficients previously computed at time  $t_i$ . They also provide an error estimate, which depends on the grid spacing used to interpolate the density function. This error is found to be quadratic in spatial discretization.

In this paper we propose a new methodology which belongs to the area of quadrature algorithms. The method, we have called perturbative moment expansion (PME), focuses the attention on distribution moments instead of PDF's. The main idea is based on two legs:

- (a) a generic PDF can be always reconstructed, knowing its moments, by resorting to an extension of Gram-Charlier Series [22]. More precisely a PDF can be expressed as a perturbative series expansion around another (simpler) PDF, by matching all moments (up to a given order) of the original distribution. These arguments are presented in section 2.
- (b) on the other hand, basic operations involving PDF's, are simpler in terms of moments. For instance, in section 3, we show how the convolution product between two distributions, can be reduced to simple arithmetics by reformulating the problem through moments.

Because often the option pricing can be reduced to a recursive numerical integrations over density functions, PME permits to solve efficiently the problem with little computational efforts (section 4). Indeed by including in the calculation just the first four moments (mean, volatility, skewness and kurtosis) the option pricing accuracy could be already high.

In particular in section 4 we show how to implement our method for a variety of contingent claims, that is: barrier, asian and reverse cliquet options.

Finally section 5, is devoted to discuss briefly some conclusions and remarks.

## 2 Perturbative moments expansion of a probability distribution function

Among quadrature methods for option pricing, a key problem is to model accurately PDF. In this section we will show how it is possible to accomplish the task in a very efficient way.

Let us consider a generic stochastic variable  $x$  with PDF given by  $P$ . The  $P$  moment of order  $k$ , is defined as:

$$\langle [x - \langle x \rangle_P]^k \rangle_P = \int_{-\infty}^{+\infty} (x - \langle x \rangle_P)^k P(x) dx . \quad (1)$$

where  $\langle x \rangle$  is the mean of  $P$ :  $\langle x \rangle_P = \int x P(x) dx$ .

The normalized moment of order  $k$ , is defined as follows:

$$\mu_P^{(k)} = \frac{\langle [x - \langle x \rangle_P]^k \rangle_P}{\{\langle [x - \langle x \rangle_P]^2 \rangle_P\}^{\frac{k}{2}}} . \quad (2)$$

By indicating with  $\sigma_P$  the square root of second moment (i.e. the standard deviation of  $P$ ):  $\sigma_P = \{\langle [x - \langle x \rangle_P]^2 \rangle_P\}^{\frac{1}{2}}$ , we can always decompose  $x$  as:

$$\begin{aligned} x &= \langle x \rangle_P + \sigma_P y , \\ P(x) &= \frac{1}{\sigma_P} \bar{P} \left( \frac{x - \langle x \rangle_P}{\sigma_P} \right) , \end{aligned} \quad (3)$$

where  $y$  is a stochastic variable with zero mean and unit variance and  $\bar{P}$  denotes its PDF.

It turns out that the probability density function  $\bar{P}$ , can be represented as a series expansion in terms of a polynomial multiplied by the normal density  $\Phi_{0,1}$  (with unit variance and zero mean). The polynomial accounts, in this way, the departure of the original PDF from normality. This expansion is known in literature as Gram-Charlier Series [22]. Usually the infinite series is truncated to a given order, very often up to fourth moment, while higher moment corrections are neglected. In such a way it is possible to incorporate adjustments in the probability distribution for non-normal skewness and kurtosis effects (the former being the third moment and accounts for asymmetric tails, while the later corresponds to the fourth moment and incorporates, for value higher than three, fatness in the tails).

In past years some authors [25, 26, 27] have resorted to Gram-Charlier Series to derive corrections to Black & Scholes formula for plain vanilla options,

by including skewness and leptokurticity effects in the distribution of stock returns.

The Gram-Charlier Series, truncated to the fourth moment, reads:

$$\bar{P}(x) = \left[ 1 + \frac{\mu_3}{6} H_3(x) + \frac{\mu_4 - 3}{24} H_4(x) + \dots \right] \Phi_{0,1}(x) , \quad (4)$$

where  $\Phi_{0,1}$  is the normal distribution with zero mean and unit variance,  $\mu_3$  and  $\mu_4$  are respectively the skewness and kurtosis of  $\bar{P}$  and  $\{H_n(x)\}_{n \in \mathcal{N}}$  denote the Hermite polynomials [1].

More generally, we can write:

$$\bar{P}(x) \approx \left( \sum_{i=0}^K c_i x^i \right) \Phi_{0,1}(x) , \quad (5)$$

where the coefficients  $c_i$  (up to order  $K$ ) can be easily computed by imposing the equivalence of the first  $K + 1$  moments of both sides of equation (5):

$$\sum_{i=0}^K c_i \langle x^{i+j} \rangle_{\Phi_{0,1}} = \langle x^j \rangle_{\bar{P}} , \text{ for: } j = 0, 1, 2, \dots, K , \quad (6)$$

where we have indicated with  $\langle x^i \rangle_{\Phi_{0,1}}$  the gaussian moment of order  $i$ . Remember that for a gaussian distribution with zero mean and unit variance:

$$\begin{aligned} \langle x^j \rangle_{\Phi_{0,1}} &= 0 && \text{if } j \text{ is odd ,} \\ \langle x^j \rangle_{\Phi_{0,1}} &= 1 \cdot 3 \cdot 5 \cdot \dots \cdot (j-1) && \text{if } j \text{ is even ,} \end{aligned} \quad (7)$$

moreover,  $\bar{P}$  is a PDF with zero mean and unit variance, therefore:

$$\langle x^0 \rangle_{\bar{P}} = 1, \langle x^1 \rangle_{\bar{P}} = 0 \text{ and } \langle x^2 \rangle_{\bar{P}} = 1.$$

By reverting the linear system equations (6), the coefficients  $\{c_i\}_0^K$  can be easily found in terms of a linear combination of the first  $K + 1$  moments of  $\bar{P}$ . Therefore, in some sense, the equation (5), can be regarded as a perturbative moment expansion (PME) of  $\bar{P}$ , around gaussian function  $\Phi_{0,1}$ . The convergence of the series to  $\bar{P}$  is guaranteed by some theorems [23].

It is straightforward to extend the equations (5) and (6), to the case where  $\Phi_{0,1}$ , is substituted with a generic PDF,  $\varphi$  (again with zero mean and unit variance). Of course, it is intuitively, that the quality of approximation in (5), at a given order  $K$ , depends on the “distance” between  $\bar{P}$  and  $\varphi$ .

### 3 The convolution of probability distributions via moments calculation

Let us consider two independent stochastic variables  $x_1$  and  $x_2$ , each of them characterized by a different probability distribution function  $P_1$  and  $P_2$ . The sum of the two variables:  $z = x_1 + x_2$  is again a stochastic variable with a probability distribution,  $P$ , given by the convolution of  $P_1$  and  $P_2$ :

$$P(z) = \int \int_{x_1+x_2=z} P_1(x_1) P_2(x_2) dx_1 dx_2 = \int_{-\infty}^{+\infty} P_1(x_1) P_2(z - x_1) dx_1. \quad (8)$$

Starting from the above equation, it is easy to find the composition law connecting the  $P$  moments to  $P_1$  and  $P_2$  moments:

$$\begin{aligned} \langle z \rangle_P &= \langle x_1 \rangle_{P_1} + \langle x_2 \rangle_{P_2} \\ \langle [z - \langle z \rangle]^n \rangle_P &= \\ &= \sum_{\nu=0}^n \binom{n}{\nu} \langle [x_1 - \langle x_1 \rangle]^\nu \rangle_{P_1} \cdot \langle [x_2 - \langle x_2 \rangle]^{n-\nu} \rangle_{P_2}, \end{aligned} \quad (9)$$

On the opposite of eq. (8), the moments composition law is quite simple and straightforward. Therefore an alternative approach to compute the convolution product between  $P_1$  and  $P_2$ , in order to find out  $P$ , consists to compute  $P$  moments (up to a certain order) by using (9) and then reverting to  $P$  by considering the perturbative expansion (5). The potentiality of this technique could be appreciated by considering the following problem: given a stochastic variable  $z$  characterized by a probability distribution  $P$ , we ask to find out the PDF,  $P$ , such that the convolution:  $P_{1/2} \circ P_{1/2}$  is equal to  $P$  (in other words we are asked to extract the square root of  $P$  respect to convolution product). This problem could be hard from a numerical point of view by considering the eq. (8) but becomes quite simple by resorting to PME and moments composition law. Indeed, within PME scheme, the problem reduces to simple algebra: once we have got the moments characterizing the square root of  $P$ , via inversion of eq. (9), it is simple to reconstruct the corresponding PDF by using equation (5).

If we consider the sum of  $n$  i.i.d. variables, with distribution function  $P_1$ , the PDF of the sum, is given by  $P_n = \prod_1^n P_1$  (where  $\prod$  refers to the convolution product). In terms of moments this equation becomes:

$$\langle [x - \langle x \rangle]_n^k \rangle_{P_n} = n \langle [x - \langle x \rangle]_1^k \rangle_{P_1} +$$

$$+ \sum_{\nu=1}^{k-1} \binom{k}{\nu} \left\{ \sum_{j=1}^{n-1} \langle [x - \langle x \rangle_{P_j}]^{k-\nu} \rangle_{P_j} \right\} \langle [x - \langle x \rangle_{P_1}]^\nu \rangle_{P_1} . \quad (10)$$

Equation (10), permits to evaluate iteratively, all  $P_n$  moments. More precisely, a solution can be easily found for  $k = 2$ , then, starting from the knowledge of  $k - 1$ -moment of  $P_n$ , we have a rule to construct the next  $k$ -moment of  $P_n$ .

As an example, we can solve iteratively equation (10), for first values of  $k$  (i.e.  $k = 2$ ,  $k = 3$  and  $k = 4$ , which give the evolution laws, respectively, for volatility, skewness and kurtosis):

$$\langle [x - \langle x \rangle_{P_n}]^2 \rangle_{P_n} = n \langle [x - \langle x \rangle_{P_1}]^2 \rangle_{P_1} , \quad (11)$$

$$\mu_{P_n}^{(3)} = \frac{\mu_{P_1}^{(3)}}{n^{1/2}} , \quad (12)$$

$$\mu_{P_n}^{(4)} = 3 + \frac{\mu_{P_1}^{(4)} - 3}{n} . \quad (13)$$

More generally, starting from eq. (10), it possible to derive a perturbative series expansion of reduced moments  $\mu_{P_n}^{(k)}$ , in powers of  $\frac{1}{\sqrt{n}}$ . In equation (14), we show the first corrections beyond gaussian result ( $n = \infty$ ):

$$\begin{aligned} \mu_{P_n}^{(k)} &= \mu_{\Phi_{0,1}}^{(k+1)} \left[ \frac{k-1}{3!} \frac{\mu_{P_1}^{(3)}}{n^{1/2}} + O\left(\frac{1}{n^{3/2}}\right) \right] & \text{k odd} , \\ \mu_{P_n}^{(k)} &= \mu_{\Phi_{0,1}}^{(k)} \left[ 1 + \frac{k(k-2)}{4!} \frac{\mu_{P_1}^{(4)} - 3}{n} + O\left(\frac{1}{n^2}\right) \right] & \text{k even} , \end{aligned} \quad (14)$$

where  $\mu_{\Phi}^{(k)}$  represents the normalized gaussian moment of order  $k$ .

Formula (14), shows that for large  $n$ , the PDF,  $P_n$ , converges (according to central limit theorem) to a gaussian distribution and the first two corrections to the asymptotic limit, depends only from  $\mu_{P_1}^{(3)}$  and  $\mu_{P_1}^{(4)} - 3$  (this result is the PME version of a famous theorem obtained by Kolmogorov and Gnedenko, for probability distributions, in 1954 [24]<sup>1</sup>).

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<sup>1</sup>The theorem provide an asymptotic series expansions of distribution function  $P_n$  as:

$$P_n(x) - \Phi_{0,1}(x) = \Phi_{0,1}(x) \sum_{j=0}^{+\infty} \frac{Q_j(x)}{n^{\frac{j}{2}}} \quad (15)$$

where  $Q_j$  is a polynomial whose coefficients depend uniquely from the first  $j + 2$  moments of  $P_n$ . The explicit form of  $\{Q_j\}$  can be found in [24]

## 4 Perturbative moment expansion for pricing European style options

The option pricing of path dependent european options with discrete time monitoring, is equivalent to solve a multiple integral. As shown in literature [16, 17, 20, 18, 19, 21], often this multiple integral can be reduced to nested one-dimensional integrals (one for each fixing dates). The problem faced by quadrature methods is how to treat accurately the density functions involved in such integrals (see the introduction).

In this paper, we propose a new scheme for modeling density functions, avoiding arbitrary choices of parametric functions as well as large grids which can lead easily to high computational costs.

The basic idea relies from one hand on perturbative moment expansion of a generic PDF (equation (5)) and from the other hand on moments composition laws derived in section 3.

We expose PME technique in practice, by considering three different kind of path dependent options:

- (i) asian options with discrete fixings (chapter 4.1);
- (ii) reverse cliquet options (chapter 4.2);
- (iii) barrier options with discrete monitoring dates (chapter 4.3);

for each of them, we present results obtained with PME approach. A comparison with other popular techniques (Monte Carlo, Quasi Monte Carlo, recursive numerical integration, binomial tree etc.) is also reported.

### 4.1 Asian options with discrete fixings

#### 4.1.1 Asian options: contract description

We consider a call asian option written on a stock with initial value  $S_0$  and volatility  $\sigma$ . The interest rate,  $r$ , is assumed, for simplicity, to be a constant (indeed in our scheme this hypothesis is not necessary).

The contract pay-off is defined as:

$$\text{Pay-off}_{\text{Asian}} = \text{Max} \left( \frac{\sum_{i=0}^m S_i}{m+1} - E, 0 \right), \quad (16)$$

where  $S_i$  is the stock price at time  $t_i = i T/m$ ,  $m$  is the number of equally spaced intervals and  $T$  indicates the time to maturity.

#### 4.1.2 Asian options: PME problem formulation

Equity price at time  $t_j$ , can be written as:

$$S_j = S_0 e^{\sum_{i=1}^j (\rho_i + v_i e_i)} , \quad (17)$$

where:  $e_i$ 's are independent  $N(0, 1)$  random variables with zero mean and unit variance;  $\rho_i$  and  $v_i^2$  are respectively the mean and variance of stock log-returns calculated between  $t_{i-1}$  and  $t_i$ :

$$\begin{aligned} \rho_i &= \left( r - \frac{\sigma^2}{2} \right) (t_i - t_{i-1}) , \\ v_i &= \sigma \sqrt{t_i - t_{i-1}} . \end{aligned} \quad (18)$$

Now, we introduce the stochastic variables,  $a_i$ , defined recursively as follows:

$$a_0 = \log(1 + e^{a_1}) , \quad (19)$$

$$a_i = \rho_i + v_i e_i + \log(1 + e^{a_{i+1}}) , \quad (20)$$

$$a_m = \rho_m + v_m e_m . \quad (21)$$

$a_i$ 's can be also re-written as

$$a_i = \hat{\rho}_i + \hat{v}_i \hat{e}_i , \quad (22)$$

where  $\hat{\rho}_i$  and  $\hat{v}_i^2$  are the mean and variance of  $a_i$  and  $\{\hat{e}_i\}$  are new stochastic variables with zero mean and unit variance.  $\hat{e}_i$ 's variables, on the opposite of  $e_i$ 's, are strictly related one to each other; more important, their PDF's, (referred in the following as  $P_{\hat{e}_i}$ ), are in principle not normal.

The option value,  $c_{\text{asian}}$ , can be then calculated as:

$$c_{\text{asian}} = e^{-rT} \int_{-\infty}^{+\infty} \text{Max} \left[ \frac{e^{\hat{\rho}_0 + \hat{v}_0 x}}{m+1} - E, 0 \right] P_{\hat{e}_0}(x) dx . \quad (23)$$

Therefore, the pricing of an asian option, with discrete fixings, is reduced to the calculation of a one-dimensional integral, with a PDF given by  $P_{\hat{e}_0}$ . The problem is therefore reduced to compute recursively:  $\hat{\rho}_i$ ,  $\hat{v}_i$  and  $P_{\hat{e}_i}$  backward, until  $i = 0$  is reached. This task could be well accomplished, by using a perturbative moment expansion (PME) of  $P_{\hat{e}_i}$  around normal distribution  $\Phi_{0,1}$ . In detail, the iterative PME scheme for an asian option is defined as follows:

- (A) fix the number of moments,  $l$ , to be used in PME algorithm;
- (B) let us start from  $i = m$ , Eqs. (21) and (22) give  $\hat{\rho}_m = \rho_m$ ,  $\hat{v}_m = v_m$  and  $P_{\hat{e}_m} = \Phi_{0,1}$ ;
- (C) proceeding backward, for  $i < m$ , let us introduce a dummy stochastic variable:  $y_{i+1} = \log(1 + e^{\hat{\rho}_{i+1} + \hat{v}_{i+1} \hat{e}_{i+1}})$ . Knowing  $\hat{\rho}_{i+1}$ ,  $\hat{v}_{i+1}$  and  $P_{\hat{e}_{i+1}}$ , we can compute, numerically, all moments (up to order  $l$ ) of  $y_{i+1}$ .  
 Observing that  $a_i$  is the sum of two independent stochastic variables (i.e.  $\rho_i + v_i e_i$  and  $y_{i+1}$ ), we can calculate, by applying the moments composition rule derived in eq. (9):  $\hat{\rho}_i$ ,  $\hat{v}_i$  and all moments (up to  $l$ ) of  $P_{\hat{e}_i}$ .  
 Finally, by resorting to eq. (5), we are able to reconstruct  $P_{\hat{e}_i}$  from its moments.
- (D) when  $i = 0$ , is reached, we can compute the option value via eq. (23).

The higher the number of moments we include in the calculation, the higher will be the precision of option value estimate. However, with just including kurtosis and skewness (and neglecting higher moments corrections), we have got excellent results (the percentage error is less than 0.1 %, see tables 1 and 2).

#### 4.1.3 Asian options: PME option pricing, a comparison with other techniques

In order to compare our results with literature, we have considered two set of parameter values, reported in the captions of table 1 and 2.

In table 1 we compare PME results (number of moments,  $l$ , ranging from 4 up to 20) against other popular techniques: (i) Monte Carlo (MC) with  $N = 10^8$  scenarios, where we have resort to antithetic variables technique in order to reduce variance errors; (ii) Recursive Numerical Integration (RNI) and (iii) Mixed Density Approximation (MDA), which represents a modification of traditional RNI algorithm in which an appropriately parametrization of distribution functions is used.

In table 2, a comparison of PME with MC, RNI and binomial tree (adopting forward shooting grid algorithm) of Barraquand and Pudet [5] is also presented.

Asian option pricing with different techniques.				
	$\sigma = 5 \%$	$\sigma = 10 \%$	$\sigma = 30 \%$	$\sigma = 50 \%$
PME ( $l = 4$ )	4.30799	4.90899	8.79859	12.96664
PME ( $l = 6$ )	4.30798	4.90899	8.80149	12.97995
PME ( $l = 8$ )	4.30798	4.90899	8.80142	12.98102
PME ( $l = 10$ )	4.30798	4.90899	8.80153	12.98124
PME ( $l = 20$ )	4.30798	4.90899	8.80151	12.98097
MC	4.30795 +/- 0.00003	4.9089 +/- 0.00014	8.80095 +/- 0.00062	12.980 +/- 0.0012
RNI	4.308	4.909	8.801	12.980
MDA	4.309	4.911	8.811	12.979

Table 1: Option pricing for an asian option with:  $S_0 = 100$ ,  $r = 9 \%$ ,  $T = 1$  year,  $E = 100$  and  $m = 52$  (number of intervals). The table shows option values computed with different techniques: PME - Perturbative Moment Expansion (with different values of  $l$ ), MC - Monte Carlo ( $10^8$  scenarios with antithetic variable technique), RNI - Recursive Numerical Integration (results are due to Lim [19]) and MDA - Mixed Density Approximation (results are due to Lim [19])

Finally, in figure 1, we show the relative error (in percent) between PME and Quasi Monte Carlo (QMC) results<sup>2</sup>. QMC estimates are based on  $N = 2^{27} - 1$  simulation series, making use of Sobol low-discrepancy sequences.

The results obtained, show that PME method give excellent results (error being less than 1%) also in the case we have retained few moments ( $l = 4$ , i.e. just kurtosis and skewness).

The main advantage of PME respect to other numerical techniques are:

- (i) PME method, as all quadrature techniques, does not suffer from increasing fixing observations number or volatility values where, on the

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<sup>2</sup>In spite of the fact that QMC method does not permit to estimate the error, we have preferred this algorithm to simple MC because a lower number of scenarios are required to obtain an estimate of a given precision.

Asian options pricing with different techniques.			
	$\sigma = 10 \%$	$\sigma = 20 \%$	$\sigma = 40 \%$
PME ( $l = 4$ )	1.845289	2.920988	5.14583
PME ( $l = 6$ )	1.845298	2.921097	5.14677
PME ( $l = 8$ )	1.845297	2.921095	5.14677
PME ( $l = 10$ )	1.845298	2.921097	5.14678
PME ( $l = 20$ )	1.845298	2.921097	5.14678
MC	1.84517 +/- 0.00008	2.9209 +/- 0.00018	5.1463 +/- 0.00040
RNI	1.845	2.921	5.146
FSG	1.869	2.960	5.218

Table 2: Option pricing for an asian option with:  $S_0 = 100$ ,  $r = 10 \%$ ,  $T = \frac{91}{365}$  year,  $E = 100$  and  $m = 91$ . The table shows option value computed with different approaches: PME - Perturbative Moment Expansion (with different values of  $l$ ), MC - Monte Carlo ( $10^8$  simulations with antithetic variable technique), RNI - Recursive Numerical Integration (results are due to Lim [19]), FSG - Forward Shooting Grid (Barraquand and Pudet in [5]).

opposite, the precision of MC estimates deteriorates when  $\sigma$  or  $m$  increase.

- (ii) Once the final PDF is obtained, option prices can be easily computed for any  $S_0$  and strike price  $E$ , moreover that gives the opportunity to compute immediately the delta greek of the option.
- (iii) Respect to the methodology used in [19], PME does not require to guess or make any hypothesis about the PDF form, nor to compute PDF on a grid of points. Indeed we have only to compute the first moments (e.g. for  $l = 4$ : mean, variance, kurtosis and skew, i.e. just four numbers) at each fixing date, making the option evaluation quite fast. For a general discussion about the advantage of PME algorithm, we remand to section 5.

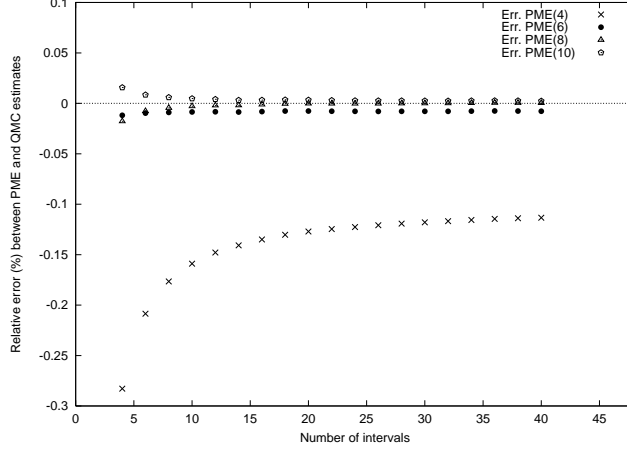


Figure 1: Percentage error between QMC result ( $N = 2^{27} - 1$ ) and PME results ( $l = 4 \div 10$ ) for an asian option with:  $S_0 = 100$ ,  $r = 9\%$ ,  $T = 1$  year,  $E = 100$ ,  $\sigma = 50\%$  and  $m$  (fixing frequency) ranging from 4 up to 40.

## 4.2 Reverse cliquet options

### 4.2.1 Reverse cliquet options: contract description

A reverse cliquet option is a product that typically have a maximum and minimum pay-out, which depends on the sum of negative performances of the underlying asset.

A typical pay-out might be written as:

$$\text{Pay-off}_{\text{Rev. cliquet}} = \text{Max} \left[ L, H + \sum_{i=1}^m \text{Min} \left( \frac{S_i - S_{i-1}}{S_{i-1}}, 0 \right) \right], \quad (24)$$

where the cliquet has  $m$  periods,  $S_i$  is the underlying price at the end of the  $i$ 'th period ( $t_{i-1}, t_i$ ), the strike level for the first period being  $S_0$  (i.e. the spot price).  $L$  is the global floor (usually set to zero) and corresponds to the minimum amount the investor will receive at the expiration date.  $H$  is the maximum coupon payable by the option.

The risk profile embedded in this option is clear: the investor is long the skew (if the skewness of stock price returns increases, it is less probable to have negative performances and therefore the contract value increases) and short volatility (indeed if volatility grows the contract price declines). As we will see, the PME approach permits to calculate at each step the effective volatility and skewness of the sum of the underlying negative performances

(also in presence of non log-normal process), making more clear the risk profile.

#### 4.2.2 Reverse cliquet options: PME problem formulation

The problem formulation in terms of perturbative moment expansions, becomes in this particular case quite simple.

Let us introduce the stochastic variables,  $R_i = \text{Min} \left( \frac{S_i - S_{i-1}}{S_{i-1}}, 0 \right)$ , i.e. the minimum value between stock performance and zero. If we indicate with  $P_i$  the PDF of variables  $R_i$ , the density function of the sum of all negative performances  $(\sum_{i=1}^m R_i)$ , is given by:

$$P = \prod_{i=1}^m P_i, \quad (25)$$

where the symbol,  $\prod$ , refers to convolution product.

The equation (25), which is indeed a multiple integral, can be simplified, by reformulating the problem in terms of PDF's moments (see section 3). More precisely, the  $P$  moments can be computed by combining, iteratively, the  $P_i$  moments via equation (9). On the other hand, the  $P_i$  moments can be easily calculated as follows:

$$\begin{aligned} R_i(x) &= \text{Min} \left[ e^{-\left(r - \frac{\sigma^2}{2}\right) \delta t_i + \sigma \sqrt{\delta t_i} x} - 1, 0 \right], \quad x \sim N(0, 1) \\ \langle R_i \rangle_{P_i} &= \int R_i(x) \Phi_{0,1}(x) dx, \\ \langle [R_i - \langle R_i \rangle_{P_i}]^k \rangle_{P_i} &= \int [R_i(x) - \langle R_i \rangle_{P_i}]^k \Phi_{0,1}(x) dx. \end{aligned} \quad (26)$$

Observe that the above integrals, can be easily “solved” in terms of a sum of cumulative normal functions.

In equation (26),  $\Phi_{0,1}(x)$  represents the normal density distribution with zero mean and unit variance and  $\delta t_i = t_i - t_{i-1}$ .

Once the  $P$  moments have been calculated, by resorting to eq. (5), it is possible to reconstruct the corresponding probability distribution function as a series expansion around normal density,  $\Phi_{0,1}$ . Then, the option price can be easily calculated as:

$$\text{option price}_{\text{Rev. Cl.}} = e^{-rT} \text{Max} \left[ L, \int (H + x) P(x) dx \right]. \quad (27)$$

Note that:

- (i) if time intervals are constant the variables  $R_i$  are i.i.d. and therefore their moments can be computed once, making the option price evaluation quite fast (the laws governing moments evolution for i.i.d. permits to compute  $P$  moments by simple arithmetics, see equations (10), (12) and (13).
- (ii) the inclusion of kurtosis and skewness effects in the process governing stock price variations does not require any modification to the algorithm, indeed it would be sufficient to substitute in eq. (26) to  $\Phi_{0,1}$  the PDF of a non log-normal process.
- (iii) by increasing the number of intervals, according to equation (14), the PDF describing the sum of negative performances, converges to a gaussian distribution (see eq. (14)). We can expect therefore, that the Gram-Charlier series truncation, used in eq. (27) to model  $P(x)$ , will become asymptotically exact for large value of  $m$ .

#### 4.2.3 Reverse cliquet options: PME option pricing, a comparison with other techniques

In table 3, we present PME results compared with QMC for different numbers of equally spaced intervals,  $m$ , (ranging from 4 up to 36). The time intervals are maintained fixed to 1/12 year. In order to make results comparable, we have considered  $H = m h$  ( $h$  constant) and option prices (in %) rescaled by a factor  $1/m$ .

PME results reported in table 3, show an excellent agreement with QMC estimates, at least for  $m \geq 12$ , even retaining only few moments in the perturbative expansion (the error is less than 0.1 % with just the first four moments).

In figure 2 we plot relative error between PME results ( $l$ , ranging from 6 up to 12) and QMC for different values of  $m$ , keeping the time interval constant. As we have stated in previous paragraph, the PDF of sum of negative performances converges, in the limit  $m \rightarrow \infty$ , to a gaussian distribution, making PME asymptotically exact. As a consequence the agreement between PME and QMC becomes better and better increasing the number of intervals  $m$ .

Reverse cliquet options pricing with different techniques.				
	$m = 4$	$m = 12$	$m = 24$	$m = 36$
PME ( $l = 4$ )	1.41681 %	1.01701 %	0.82935 %	0.72502 %
PME ( $l = 6$ )	1.42984 %	1.01755 %	0.82890 %	0.72465 %
PME ( $l = 8$ )	1.43915 %	1.01896 %	0.82914 %	0.72470 %
PME ( $l = 10$ )	1.43730 %	1.01769 %	0.82891 %	0.72467 %
PME ( $l = 20$ )	1.43417 %	1.01798 %	0.82898 %	0.72469 %
QMC	1.4336 %	1.0179 %	0.82898 %	0.72469 %

Table 3: Option pricing for a reverse cliquet option with:  $S_0 = 100$ ,  $r = 9\%$ ,  $\sigma = 30\%$ ,  $\delta_t = t_i - t_{i-1} = 1/12$  year,  $L = 0$  and  $m = 4, 12, 24, 36$ . In order to make results comparable, we have considered  $H = m h$  ( $h = 4\%$ ) and option prices (expressed in %) have been rescaled by a factor  $1/m$ . The table shows option values computed with different techniques: PME - Perturbative Moment Expansion (with different values of  $l$ ) and QMC - Quasi Monte Carlo ( $2^{27} - 1$  simulations).

### 4.3 Barrier options with discrete monitoring dates

#### 4.3.1 Barrier options: contract description

The pay-off of a generic knock out barrier option can be written as follows:

$$\text{Pay-off}_{\text{Barrier}} = \begin{cases} \mathcal{F}[S(T)] & \text{if } S(t_i) > B_i \text{ for all observation dates } \{t_i\} \\ b & \text{otherwise} \end{cases} \quad (28)$$

where:  $\mathcal{F}$  is a generic positive function which represents the option pay-off if the barrier has not been touched,  $b$  is known as “rebate” and represents the amount paid by the contract if the barrier has been touched (often  $b = 0$ ),  $B_i$  is the barrier level at time  $t_i$ <sup>3</sup>,  $\{t_i\}_{i=1,2,\dots,N}$  are the discrete observation dates set,  $S_0$  the spot price,  $S(t_i)$  the underlying price at time  $t_i$ .

<sup>3</sup>For constant barrier level  $B_i = B \quad \forall i$ , in the present discussion we do not make any assumption about  $\{B_i\}$ .

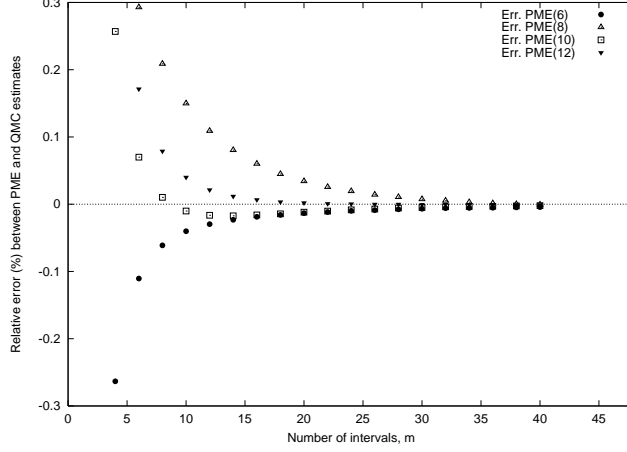


Figure 2: Reverse cliquet option: the graph shows the relative error between PME and QMC estimates, versus the number of intervals,  $m$ . Different values of  $l$  are considered (ranging from 6 up to 12). Data refers to:  $S_0 = 100$ ,  $r = 9\%$ ,  $\delta_t = t_i - t_{i-1} = 1/12$  year,  $\sigma = 30\%$ ,  $L = 0$ ,  $H = mh$  with  $h = 4\%$  and  $m =$  ranging from 4 up to 40.

#### 4.3.2 Barrier options: PME problem formulation

The problem implicit in a barrier option can be reformulated in terms of PDF's. The stock price at time  $t_i$  is given by:

$$S(t_i) = S_0 e^{\sum_{j=1}^i \left( r - \frac{\sigma^2}{2} \right) \delta t_j + \sigma \sqrt{\delta t_j} e_j}, \quad (29)$$

where  $\delta t_j = t_j - t_{j-1}$  and  $\{e_j\}$  are independent normally distributed random variables with zero mean and unit variance.

Let us define the following stochastic variables:

$$z_i = \left( \log \frac{S(t_i)}{S_0} \mid S(t_j) > B_j \quad \forall j \leq i \right), \quad (30)$$

which embedding the condition that all observations until time  $t_i$  are above the barrier level.

Let us introduce:

$$p_i = \text{probability that: } S(t_j) > B_j \quad \forall j \leq i, \quad (31)$$

i.e. the probability that the option is alive at time  $t_i$ .

Then, the option price can be expressed in terms of PDF of  $z_m$ ,  $P_{z_m}$ , as:

$$\text{option price}_{\text{KO}} = e^{-r(t_m-t_0)} \left[ (1-p_m) b + p_m \int \mathcal{F}(S_0 e^x) P_{z_m}(x) dx \right]. \quad (32)$$

The stochastic variables  $\{z_i\}$  can be recursively related one to each other, by the following equations:

$$\begin{aligned} z_0 &= 0, \\ w_i &= z_{i-1} + \left( r - \frac{\sigma^2}{2} \right) \delta t_i + \sigma \sqrt{\delta t_i} e_i, \\ z_i &= (w_i \mid S(t_i) > B_i) = \left( w_i \mid w_i > \log \frac{B_i}{S_0} \right), \end{aligned} \quad (33)$$

where we have introduced the dummy variables  $w_i$ .

The equations set (33) can be translated in terms of PDF's as follows:

$$P_{z_0}(x) = \delta(x), \quad (34)$$

$$P_{w_i} = P_{z_{i-1}} \circ \Phi_{(r-\sigma^2/2)\delta t_i, \sigma^2 \delta t_i}, \quad (35)$$

$$P_{z_i}(x) = \begin{cases} \frac{1}{k_i} P_{w_i}(x) & \text{if } x > \log \frac{B_i}{S_0} \\ 0 & \text{otherwise} \end{cases} \quad (36)$$

where:  $\delta(x)$  is the Dirac delta distribution, “ $\circ$ ” indicates the convolution product,  $\Phi_{\rho, v^2}$  is the normal distribution with mean  $\rho$  and variance  $v^2$  and  $k_i$  is the probability to find  $w_i$  below  $\log \frac{B_i}{S_0}$ :

$$k_i = \int_{\log \frac{B_i}{S_0}}^{\infty} P_{w_i}(x) dx. \quad (37)$$

In this framework, the probabilities  $p_i$  are then related by the recursive relation:

$$p_i = p_{i-1} \cdot k_i. \quad (38)$$

The PME iterative scheme for a barrier option evaluation can be therefore structured as follows:

- (A) fix the number of moments,  $l$ , to be used in PME algorithm;
- (B) repeat, starting from  $i = 0$ , the steps B1–B3 until  $i = m$  is reached:

(B1) if  $i = 0$ ,  $P_{z_0}$  is the Dirac delta distribution, therefore its moments, at any order are identically zero apart from moment of order zero which is, by definition, equal to 1.

Otherwise for  $i > 0$ : knowing the moments of  $P_{z_{i-1}}$ , compute all moments, up to order  $l$ , of PDF  $P_{w_i}$  by using equation (9)<sup>4</sup>.

(B2) by reverting to equations (3), (5) and (6), reconstruct PDF  $P_{w_i}$  by a Gram Charlier series around normal density function  $\Phi_{0,1}$ .

(B3) given  $P_{w_i}$ , from eq. (36), compute all moments of  $P_{z_i}$  up to order  $l$ .

(C) Reached  $i = m$ , we know all  $P_{z_m}$  moments up to  $l$ , again, by using equations (3), (5) and (6) we can develop  $P_{z_m}$  as a Gram Charlier series around normal density. Once that is done, the option price can be easily evaluate by means of eq. (32).

It interestingly to note that within the scheme presented above, we need to evaluate only integrals of the form:

$$I_n(d) = \frac{1}{\sqrt{2\pi}} \int_{-d}^{+\infty} x^n e^{-\frac{1}{2}x^2} dx, \quad (39)$$

which can be reduced to the evaluation of the well known inverse cumulative normal function<sup>5</sup>. Therefore we do not need any numerical methods to compute the required integrals.

A last remark on PME scheme: if we want to treat also non log-normal processes for equity prices evolution, the only modification to be implemented regards the equations (33) and (35), where in place of  $e_i$  and  $\Phi_{0,1}$  we must consider a non normal stochastic variable/PDF. Hence, when computing the

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<sup>4</sup>The moments of a gaussian distribution are known at every order, see equation (7).

<sup>5</sup>For instance:

$$\begin{aligned} I_0(d) &= \text{Err\_f}(d), \\ I_1(d) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2}, \\ I_2(d) &= \text{Err\_f}(d) - d \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2}, \\ I_3(d) &= (2 + d^2) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2}, \\ &\vdots \end{aligned}$$

$P_{w_i}$  moments via eq. (9), we just need to consider the real PDF moments, different from those of a normal distribution. As an example in figure 3, we report how the option price changes when fat tails (i.e. non-normal kurtosis) are considered. As expected, increasing leptokurticity of stock returns

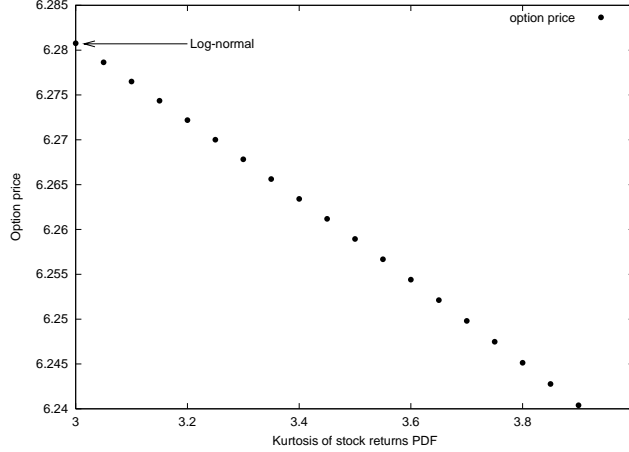


Figure 3: Price of a down-out call option, calculated with PME ( $l = 20$ ) by considering a leptokurtic behavior (i.e. kurtosis greater than 3) in probability density function of stock returns.  $S_0 = 100$ ,  $r = 10\%$ ,  $T = 0.2$  year,  $E = 100$ ,  $\sigma = 30\%$ ,  $B = 89$  and  $m = 5$ .

distribution has the effect to enlarge the chances of touch the barrier before maturity, lowering therefore the option value.

#### 4.3.3 Barrier options: PME option pricing, a comparison with other techniques

In order to make a quantitative comparison with other numerical methods, we have considered a down-out barrier option with constant barrier  $B$ . The pay-off at maturity is given by:

$$\text{Pay-off} = \begin{cases} S - E & \text{if } S > E \\ 0 & \text{otherwise} \end{cases} \quad (40)$$

i.e. the usual call plain vanilla pay-off.

For this kind of option, recently Fusai et al. [3] have derived a closed form solution. Table 4 shows, for different values of barrier level,  $B$ , and observation dates number,  $m$ , a comparison of PME with the exact result and other numerical techniques.

Barrier option pricing with different techniques.								
<b>B</b>	89	95	97	99	89	95	97	99
<b>m</b>	5	5	5	5	25	25	25	25
PME(4)	6.27407	5.68309	5.17957	4.49928	6.20762	5.11871	4.14616	2.82893
PME(12)	6.27954	5.67106	5.16749	4.48947	6.19971	5.08100	4.11596	2.81157
PME(20)	6.28077	5.67109	5.16726	4.48918	6.20850	5.08032	4.11508	2.81154
PME(32)	6.28076	5.67110	5.16725	4.48917	6.21032	5.08096	4.11559	2.81224
Exact	6.28076	5.67111	5.16725	4.48917	6.20995	5.08142	4.11582	2.81244
QMC	6.28075	5.67111	5.16726	4.48912	6.210005	5.08156	4.11561	2.81233
NRI	6.2763	5.6667	5.1628	4.4848	6.2003	5.0719	4.1064	2.8036
CMF	6.284	5.646	5.028	4.050	6.210	5.084	4.113	2.673
TT	6.281	5.671	5.167	4.489	6.21	5.081	4.115	2.812
SRQM	6.2809	5.6712	5.1675	4.4894	6.2101	5.0815	4.11598	2.8128

Table 4: Option pricing for a barrier option with:  $S_0 = 100$ ,  $E = 100$ ,  $r = 10$  %,  $T = 0.2$  year,  $\sigma = 30$  %. The table shows option price computed with different techniques: PME ( $l = 4, 12, 20$  and  $32$ ), Exact results due to Fusai [3], QMC - Quasi Monte Carlo method (calculated with  $2^{27} - 1$  scenarios), NRI - Numerical Recursive Integration (results are due to Aitsahlia [16]), CMF - Continuous Monitoring Formula (Broadie in [8]), TT - trinomial tree (results are due to Broadie [9]) and SRQM - Simpson Recursive Quadrature Method (Fusai in [21]).

The agreement is quite good and it is better, in general, than the corresponding result achievable by simple NRI, at least when the number of observation dates is relatively small (as it is in the example considered in table 4).

In figure 4, we report the relative percentage error between PME estimates and the exact result, for two different monitoring frequencies. The graph shows the accuracy of the PME method, which is already good (errors less than 0.1 %) just including few moments.

However the number of moments to be considered in order to reach a good precision is higher respect to asian options (see chapter 4.1) or reverse cliquet

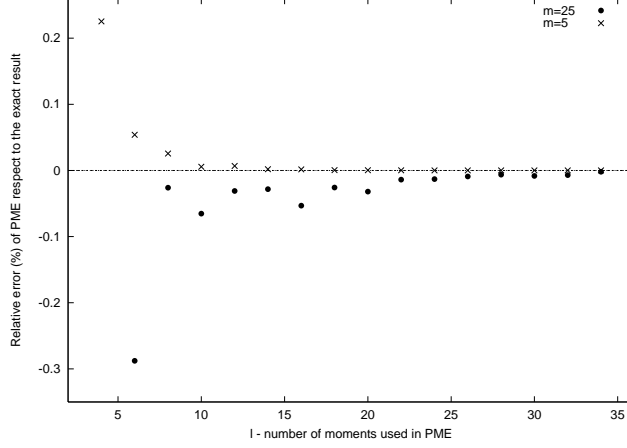


Figure 4: Percentage error between PME estimates ( $l = 4, 6, \dots, 34$ ) and exact result for a down-out call option with  $S_0 = 100$ ,  $r = 10\%$ ,  $T = 0.2$  year,  $E = 100$ ,  $\sigma = 30\%$ ,  $B = 99$  and  $m = 5, 25$ .

options (see chapter 4.2). The reason being the strong skewness (especially when equity spot price is near to the barrier or  $m$  is large) characterizing the underlying probability distribution,  $P_{z_m}$ , which leads to a strong departure from gaussian behavior. Indeed large skewness values lead to a Gram-Charlier expansions which may not be non-negative, especially on the distribution tails (compromising therefore the acceptance of the function as a true density). Because in a barrier option a key point is to measure the probability to touch the barrier at each step (i.e. to calculate accurately an integral on the left tail) the non positive definiteness of the PDF's tails, may lead to errors that compromise the pricing quality. That requires, especially when  $m$  is large, a correction to simple PME scheme, in order to fix (at least partially) the problem. A simple trick consists to substitute the step (B2) of the algorithm described in the previous paragraph, by a two layer procedure:

(B2)

- (L1) reconstruct PDF,  $P_{w_i}$ , as a perturbative series expansion around normal density:  $P_{w_i} = \left( \sum_{i=0}^K c_i x^i \right) \Phi_{0,1}(x)$  ;
- (L2) (a) find the lower negative value  $x_0$  such that:  $\sum_{i=0}^K c_i x_0^i = 0$ ;  
(b) considering a new function  $\hat{P}_{w_i}(x) = \begin{cases} P_{w_i}(x) & \text{if } x \geq x_0 \\ 0 & \text{otherwise} \end{cases}$

Due to the truncation procedure, we can expect that  $\hat{P}_{w_i}$  moments depart slightly from initial target moments. However we can apply again the perturbative expansion procedure using as basic PDF the  $\hat{P}_{w_i}$  function <sup>6</sup> instead of a simple gaussian.

In that way it is possible to derive an improved perturbative series expansion, where the problem of non positiveness is partially fixed <sup>7</sup>. In table 5, we report PME valuations (using such improved algorithm and including the first twelve moments,  $l = 12$ ) versus exact results. The error is typically lower than 0.1 %, also for large number of observation dates,  $m$ .

Barrier options pricing (improved PME / exact results)			
m	PME(12)	Exact result	Relative error
10	4.1820	4.18224	0.006 %
50	3.1260	3.12633	0.011 %
80	2.9391	2.93918	0.003 %
100	2.8643	2.86442	0.004 %
120	2.8087	2.80903	0.012 %
150	2.7461	2.7474	0.047 %
180	2.7020	2.70163	-0.014 %
200	2.6746	2.67682	0.083 %
220	2.6531	2.65545	0.089 %
250	2.6253	2.628099	0.107 %
280	2.6018	2.60534	0.136 %
300	2.5879	2.59056	0.103 %
500	2.5018	2.50259	0.032 %

Table 5: Option pricing for a barrier option with: spot price = 100, strike = 100, barrier = 98,  $r = 10$  %,  $T = 0.2$  year,  $\sigma = 30$  %. The table shows a comparison between the option value computed with an improved PME technique ( $l = 12$ ) against the exact result (Fusai et al. in [3]) for different  $m$  values.

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<sup>6</sup>Before that, it is necessary to rescale  $\hat{P}_{w_i}$ , according to eq. (3), in order to obtain a PDF with zero mean and unit variance.

<sup>7</sup>Indeed, the above procedure does not guarantee that the new expansion is non negative for any value of  $x$ . However numerical evidence shows a good improvement and a significantly reduction of the non positiveness problem.

## 5 Summary and Conclusions

In this article we have proposed a new methodology for option pricing belonging to quadrature techniques. Respect to other quadrature algorithms, where for instance density functions are modeled through a grid of points or polynomial interpolations, our method adapts a GramCharlier series expansion around a given distribution (usually a gaussian function).

The highlights features of the method are the following:

- (i) by representing PDF's through their moments, we are able to capture all the essential features of the problem by using just few parameters, improving computational performances and memory usage. How we have shown in the paper, the crude choice of retaining only the first four moments (i.e. kurtosis and skewness) in PME scheme, provides an excellent approximation of the option value (with errors less than 0.1% for asian and reverse cliquet options). Furthermore, by increasing the number of moments retained in PDF expansion, the numerical solution converges to the exact result, making possible, in principle, to increase progressively the estimate precision.
- (ii) The method is extremely simple and natural from a conceptual point of view and therefore easily implementable.
- (iii) It is extensible to different pay-off contracts (indeed the programming code needs very few modifications changing the contract features).
- (iv) non log-normal processes (i.e. stochastic processes characterized by a non log-normal PDF), can be naturally treated within a PME scheme, without any modification to the algorithm.

Moreover, the proposed method, maintains all the typical advantages of quadrature methods, that is:

- (i) we need to perform computations only at trigger times;
- (ii) the CPU time scales linearly with the number of observation dates.
- (iii) the price estimate is not too sensitive to changes in volatility;
- (iv) there are no time discretization errors.
- (v) it is easily to incorporate additional exoticity in the contract (e.g. for a barrier option, it is straightforward to include a time varying barrier).

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